Separable Potentials in the Complex *l* Plane

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The analytic behavior of a certain kind of nonlocal separable potentials previously considered by one of us (ANM) is studied in the complex angular momentum plane. The amplitude derived from such potentials has cuts only in the locations expected for the Mandelstam representation, including one corresponding to the crossed channel. The spectral functions are explicitly evaluated. A study of the singularities in the complex l plane of the partial wave amplitudes shows that there is only one Regge trajectory to the right of the $\mathrm{Re}\, l=-3/2$, and that its behavior is that of the principal Regge trajectory corresponding to a Yukawa potential. This is confirmed through the evaluation of the high-energy limit of the total amplitude in the crossed channel.

1. INTRODUCTION

AMONG the successes of the Regge formalism, perhaps the most important from the physical point of view is the role of the "trajectories" in the interpretation of various resonances and high-energy cross sections. The extrapolation to the relativistic domain of the ideas derived from potential scattering has been proposed in detail by Chew,[1,2] Frautschi,[1–3] Gell-Mann,[3] and Gribov and Pomeranchuk.[4] Since, however, these ideas in the relativistic domain cannot easily be "proved" by conventional techniques of, say, field theory, faith in these has to be sustained, apart from general physical considerations, by the validity of the Regge formalism in potential scattering.

From the mathematical point of view, the most important success of the Regge formalism is the facility with which it is now possible to study the analyticity properties of amplitudes simultaneously in the energy (s) and momentum transfer (t) variables. The assumption of bounded behavior as $t\rightarrow\infty$, which was needed for proofs of the Mandelstam representation[5] before the complex l plane was available, strongly points to the advantages of the Regge formalism. The techniques of complex angular momentum have vastly extended the mathematical tools for handling potentials more complicated than the Yukawa type, so as to examine what kind of analytic properties obtain for the amplitudes from such potentials.

These techniques are being increasingly used for studying not only the amplitudes arising from scattering by specific types of potentials, but those from suitable truncation schemes in field theory. Thus, Lee and Sawyer[6] have found that the solution of the ladder approximation to the so-called Bethe-Salpeter equation for scattering of two spinless particles admits of Regge poles in a fashion strongly reminiscent of the corresponding behavior of the scattering amplitude from a Yukawa potential. For a more impressive example of the appearance of Regge trajectory in field theory, Frautschi,[7] and Lévy[8] independently, have shown by considering a suitable set of diagrams that the high-energy cross section for electron scattering with radiative corrections to *all orders* has exactly the same form as demanded by Regge on the basis of potential scattering alone.

Further generalizations for potential scattering are also under way. Cornwall and Ruderman[9] have shown the validity of the Mandelstam representation as well as the "correct" behavior of Regge trajectories for a class of energy-dependent, Yukawa-type potentials, signifying nonlocality in time, but not in space. On the other hand, Barut and Calogero[10] have found by studying certain types of soluble potentials (square-well and centrifugal types) that it is possible for a scattering amplitude to be meromorphic in the entire l plane without satisfying the Mandelstam representation. For such cases, those authors have found that the analytic continuation is not unique. Unless, therefore, such examples are ruled out on deeper physical grounds, e.g., crossing relations, they might well prove a stumbling block to the general acceptance of the Mandelstam representation which may remain an open question for some time before further discriminating criteria are established.

In this paper, we have been motivated by a desire to

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4 V. N. Gribov and I. Ya. Pomeranchuk, Phys. Rev. Letters **9**, 238 (1962).
5 R. Blankenbecler *et al.*, Ann. Phys. (N. Y.) **10**, 62 (1960); see also, A. Klein, J. Math. Phys. **1**, 41 (1960).
6 B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266, 2274 (1962).
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8 M. Lévy, Phys. Rev. Letters **9**, 235 (1962); see also, M. Gell-Mann and M. L. Goldberger, Phys. Rev. Letters **9**, 275 (1962).
9 J. Cornwall and M. Ruderman, Phys. Rev. **128**, 1474 (1962).
10 A. O. Barut and F. Calogero, Phys. Rev. **128**, 1383 (1962).

study the analytic properties of amplitudes from a certain class of spatially nonlocal potentials—the so-called separable potentials—with which we have been associated for some time.[11–13] It appears to us that such potentials have a very natural place in the "Regge scheme" in which the basic complex variables are s and l, in much the same way as a local potential of the Charap-Fubini type[14] has a place in the "Mandelstam scheme" which works directly with the complex variables s and t. Therefore, insofar as the Regge formalism permits a study of the analytic behavior of amplitudes in s and t via the (s,l) scheme without the usual restriction to the Lehmann ellipse, it was thought worthwhile to study this question for certain types of separable potentials which can be defined for general l values so as to fit in within the Regge framework. In a sense, the present work may be regarded as a direct continuation of an earlier paper[11] where a class of separable potentials was found to satisfy the various partial wave dispersion relations (as well as correct threshold behavior), but the analytic properties in s and t of the complete amplitude were not investigated for them. The present work now seeks to fill this gap as well as to study the predictions of such potentials with regard to the S-matrix poles.

We would like to suggest that our purpose in studying such potentials for the analytic properties of amplitudes is not entirely academic. It has been shown elsewhere[12,13] that such potentials can be profitably used for solving problems involving three-body systems, and obtaining structures for the explicit three-particle amplitudes. Now a three-body system seems to be too general an entity to be studied in an entirely kinematical fashion, so that if the analytic properties of such systems in all the relevant variables have to be understood in greater details, additional input information, preferably of dynamical origin, must be supplied in advance. This last requirement can be met, e.g., by allowing a three-particle system to satisfy the formal Schrödinger equation whose solution (if it can be expressed in a simple form) would explicitly incorporate many of the analytical properties of the former which might otherwise be hard to discover. This use of separable potentials in effecting the solution of such a Schrödinger equation has been already discussed in references 12 and 13. It may, therefore, be of interest to study the analytic properties of three-body systems with separable potentials which have been so devised as to reproduce certain "known" features of two-particle amplitudes. For example, if (as is generally believed) a two-particle amplitude satisfies the Mandelstam representation, then a separable potential devised to satisfy this crietrion can be used to construct an "exact" three-particle amplitude. It is likely that the analytic properties of such an amplitude in *all* the relevant variables would be specified in much greater details than might be afforded with the mere input information of Mandelstam representation and unitarity for the two-particle amplitudes. In any case it may be of some interest to compare the knowledge of cuts and singularities of a three-particle amplitude obtained in this manner with a corresponding knowledge derived from alternative methods, e.g., the Landau-Cutkosky[15] techniques.

[11] A. N. Mitra, Phys. Rev. **123**, 1892 (1961).
[12] A. N. Mitra, Nucl. Phys. **32**, 529 (1962).
[13] A. N. Mitra, Phys. Rev. **127**, 1342 (1962).
[14] J. Charap and S. Fubini, Nuovo Cimento **14**, 540 (1959).

In this paper we confine ourselves to a study of separable potentials in the context of two-particle amplitudes only. Study of three-particle amplitudes will be the subject of a subsequent publication.

In Sec. 2, the analytic behavior of the scattering amplitude $f(s,t)$ is studied by a generalization of the results of reference 11 to the complex l plane. It is found that $f(s,t)$ has cuts only in the locations expected for the "Mandelstam representation," with a "crossed-channel" cut in addition to the conventional one in Yukawa-potential scattering.[5] In Sec. 3, the properties of the partial wave amplitude $A_l(s)$ in the complex l plane as well as the high-energy behavior of $f(s,t)$ are investigated. The results are very similar to those obtainable from Yukawa-type potentials.

In Sec. 4 a comparison is made of the present approach with some contemporary ones, and a general procedure for constructing separable potentials from arbitrary amplitudes is suggested.

2. DOUBLE DISPERSION REPRESENTATION

We start by summarizing the main results obtained in reference 11 which is referred to as A in the following. The "potential" under consideration is defined by

$$(\mathbf{p}|V|\mathbf{p}')=-(\lambda/M)\sum_l(2l+1)v_l(p)v_l(p')P_l(\hat{p}\cdot\hat{p}') \quad (2.1)$$

where

$$v_l^2(p)=(2/p^2)Q_l(1+\tfrac{1}{2}\beta^2p^{-2}). \quad (2.2)$$

The scattering amplitude from such a potential is given by

$$f(s,t)=\sum_l(2l+1)A_l(s)P_l(1+t/2s) \quad (2.3)$$

where

$$s=k^2,\quad z=\cos\theta,\quad t=-2k^2(1-z), \quad (2.4)$$

$$A_l(s)=N_l(s)/D_l(s), \quad (2.5)$$

$$N_l(s)=(4\pi^2\lambda/s)Q_l(1+\beta^2/2s), \quad (2.6)$$

and

$$D_l(s)=1-4\pi\lambda\int_0^\infty ds'\, s'^{-1/2}Q_l(1+\tfrac{1}{2}\beta^2s'^{-1})/(s'-s). \quad (2.7)$$

As was shown in A, the amplitudes $A_l(s)$ defined by (2.5) satisfy the standard partial wave dispersion relations expected of them, with the left-hand cut in s being over $-\infty<s\leqslant-\frac{1}{4}\beta^2$, the possible poles in $-\frac{1}{4}\beta^2<s<0$, and the physical cut in $0<s<\infty$. Further,

[15] L. D. Landau, Nucl. Phys. **13**, 181 (1960); see also, R. E. Cutkosky, J. Math. Phys. **1**, 429 (1960).

the Born approximation amplitude $f_B(t)$ was shown to be exactly the same as given by the Yukawa potential,

$$V(r)=-(4\pi^2\lambda/rM)e^{-\beta r}. \tag{2.8}$$

Finally, we remark that the series (2.3) is convergent within the Lehmann ellipse. This is shown in Appendix A. Thus, our separable potential (2.2) gives at least the the same domain of analyticity over t as found for an ordinary Yukawa potential of range β, corresponding to *positive energies* ($s>0$). In addition, the results of Appendix A show that it is possible with our potential to establish the convergence of the series (2.3) even for negative s, as long as $0<t<\beta^2$. This fact will be of use later in this section in establishing the analytic behavior of $f(s,t)$ over a larger (s,t) domain, via complex angular momentum techniques.

We now turn to the investigation of further properties of the amplitude (2.3) within the Regge formalism.

Before converting (2.3) into a Regge integral, it is convenient to extract its Born approximation part explicitly. This is done by writing

$$A_l(s)=N_l(s)+B_l(s), \tag{2.9}$$

where

$$B_l(s)=N_lI_l/D_l \tag{2.10}$$

and

$$I_l(s)=1-D_l(s). \tag{2.11}$$

The sum in (2.3) due to the part $N_l(s)$ in (2.9) is found in the standard way to be

$$f_B(t)=8\pi^2\lambda(\beta^2-t)^{-1}. \tag{2.12}$$

The part of (2.3) due to $B_l(s)$ is now converted into the Watson-Sommerfeld integral representation:

$$f(s,t)=f_B(t)+\tfrac{1}{2}i\oint_C(2l+1)dl\,B_l(s)P_l(-z)/\sin\pi l, \tag{2.13}$$

where the contour C encloses the poles $l=0,\ 1,\ 2,\ \cdots$ on the real axis by going round it in the clockwise direction. Shifting the contour to the line $\mathrm{Re}l=-\frac{1}{2}$ leads to the formula

$$f(s,t)=f_B(t)+\tfrac{1}{2}i\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty}dl \times(2l+1)B_l(s)P_l\left(-1-\frac{t}{2s}\right)\Big/\sin\pi l +\sum_j(2\alpha_j+1)\beta_j(s)P_{\alpha_j}[-1-(t/2s)]/\sin\pi\alpha_j(s), \tag{2.14}$$

where $l=\alpha_j(s)$ is the jth Regge pole and $\beta_j(s)$ is the residue of $-\pi B_l(s)$ at $l=\alpha_j$:

$$\lim_{l\to\alpha_j}[l-\alpha_j(s)]B_l(s)(-\pi)=\beta_j(s). \tag{2.15}$$

The justification of the step from (2.13) to (2.14), as is now well known, depends essentially on the inequality[16]

$$|P_l(-z)/\sin\pi l|\leqslant|\pi/(2l+1)\sin\theta|^{1/2} \times\exp[(l_R+\tfrac{1}{2})|\theta_I|-(\pi-|\theta_R|)|l_I|], \tag{2.16}$$
$$(\theta=\theta_R+i\theta_I,\quad l=l_R+il_I)$$

so that for the integral in (2.14) to be convergent, one no longer needs the limitations $\xi=\theta_I<\alpha$ [as in (A9), Appendix A for integral l], but the trivial condition $\pi-|\theta_R|>0$. As a matter of fact, even for the most unfavorable case $l_R=-\frac{1}{2}$ and $\pi=\theta_R$, the convergence of the integral term in (2.14) is assured for $\mathrm{Re}s>0$ if the amplitude $B_l(s)$ goes to zero at least as fast as $(2l+1)^{-3/2}$. That this is indeed the case for our amplitude defined by (2.10) is clear from Eqs. (A4) and (A5) of Appendix A. [Convergence over the infinite semicircle in the l plane follows in the conventional manner, using (2.16) and (A5).]

To see the analytic properties of $f(s,t)$ defined by (2.14), we start by examining the physical region $s>0$ and $t<0$. Now it has been shown in Sec. 3 that there is only one Regge pole to the right of $l=-3/2$ for our potential, and that it has only the right-hand cut in s. Using this result in (2.14), we find that in the physical region ($s>0$, $t<0$), $f(s,t)$ is analytic except for the cut along the positive real axis in s. For the unphysical regions in t ($t>0$), the definition of $f(s,t)$ must be specified by analytic continuation. This is done in the conventional way by writing for $\mathrm{Re}s>0$,

$$P_l(-1-\tfrac{1}{2}ts^{-1}) =-\pi^{-1}\sin\pi l\int_0^\infty dt'\,(t'-t)^{-1}P_l(1+\tfrac{1}{2}t's^{-1}), \tag{2.17}$$

which shows that $f(s,t)$ has a cut in t along the positive real axis,[17] and that it is analytic in t for $\mathrm{Re}t<0$ and $\mathrm{Re}s>0$.

To extend this statement to negative values of s—and this is compatible with the definition (2.17)—the convergence of the Regge integral in (2.14) for negative s must first be examined. Now we have seen that the Regge form, viz. Eq. (2.14), overlaps with the series form (2.3) for $0<t<\beta^2$ and $\mathrm{Re}s>0$. However, the results of Appendix A show that (2.3) is convergent also for $\mathrm{Re}s<0$. Using the same techniques as of Appendix A for $l=-\frac{1}{2}+ip$, it is easy to show that the Regge form (2.14) certainly converges for $s=s'+i\epsilon$, $s'<-\frac{1}{2}\beta^2$, with $0<t<\beta^2$, and that this limit on s' can be pushed further to the right by a more economical manipulation of certain inequalities. Thus, we find that at least for $\mathrm{Re}s<-\frac{1}{2}\beta^2$ and $0<t<\beta^2$, the series and Regge forms are analytic and equivalent. By analytic continuation (cf., e.g., Blankenbecler *et al.*, reference 5, p. 71) therefore, the validity of (2.14) for negative s is established, and

[16] See reference 10 for a very complete discussion of the asymptotic behavior of $P_l(z)$.

[17] Actually the cut starts only from $t=\beta^2$, since it has been shown in Appendix A that the series for (2.3) is convergent, and hence analytic in t, for $t<\beta^2$, in which region the forms (2.3) and (2.13) are completely equivalent representations of $f(s,t)$.

Eq. (2.17) can now be used for analytic continuation of $P_l(-1-t/2s)$ for negative s.

The spectral representation of $f(s,t)$ in the t variable is now deduced by writing

$$f(s,t)=\pi^{-1}\int_0^\infty dt'\, A(s,t')/(t'-t), \qquad (2.18)$$

and using (2.17) to obtain (apart from subtractions)

$$A(s,t)=(2\pi)^3\lambda\delta(\beta^2-t) -\tfrac{1}{2}i\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dl\,(2l+1)B_l(s)P_l(1+\tfrac{1}{2}ts^{-1}) -(2\alpha+1)\beta(s)P_\alpha(1+\tfrac{1}{2}ts^{-1}), \qquad (2.19)$$

the last term representing the contribution of the Regge pole to the t-spectral function $A(s,t)$.[18]

To examine the analytic behavior of $f(s,t)$ in the s variable, it is once again convenient to look at Eq. (2.3) which is valid for $0<t<\beta^2$. For this range of t, the discontinuities of $f(s,t)$ defined by (2.3)–(2.12) across the real axis in the s plane are seen to be the following:

$$B_+(s,t)=\sum_0^\infty (2l+1)P_l(1+\tfrac{1}{2}ts^{-1})s^{1/2}|A_l(s)|^2, \quad (s>0); \qquad (2.20)$$

$$B_-(s,t)=\sum_0^\infty (2l+1)P_l(1+\tfrac{1}{2}ts^{-1})C_l(s), \quad (s<0), \qquad (2.21)$$

where

$$C_l(s)=(2\pi^2\lambda/s)I_l(s)D_l^{-1}(s) \times P_l(1+\tfrac{1}{2}\beta^2s^{-1})\theta(-\tfrac{1}{4}\beta^2-s), \qquad (2.22)$$

θ being the usual step function. In deriving (2.20) and (2.21), use has been made of the following relations

$$B_l(s+i\epsilon)-B_l(s-i\epsilon) =2is^{1/2}N_l^2(s)/D_l(s+i\epsilon)D_l(s-i\epsilon) =2is^{1/2}|A_l(s)|^2, \quad (s\geqslant 0); \qquad (2.23)$$

$$Q_l(x+i\epsilon)-Q_l(x-i\epsilon) =-i\pi P_l(x), \quad (-1\leqslant x\leqslant 1); \qquad (2.24)$$

$$=2i\sin\pi l\,Q_l(-x), \quad (-\infty<x<-1). \qquad (2.25)$$

Thus, apart from possible subtractions, $f(s,t)$ satisfies the s-spectral representation

$$f(s,t)=\frac{8\pi^2\lambda}{\beta^2-t}+\int_0^\infty ds'\frac{B_+(s',t)}{s'-s}+\int_{-\alpha}^{-\frac{1}{4}\beta^2} ds'\frac{B_-(s',t)}{s'-s}. \qquad (2.26)$$

It is, therefore, seen that in contrast with the s-spectral representation for a Yukawa potential,[5] our potential gives, not only the usual right-hand (unitarity) cut in the s variable, but a left-hand cut as well.[19] To extend the spectral functions $B_\pm$ beyond $0<t<\beta^2$, one must again use Regge integrals like (2.14) for (2.20) and (2.21), and obtain, apart from a Regge pole contribution to $B_+(s,t)$,

$$B_+(s,t)=\tfrac{1}{2}i\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dl \times(2l+1)s^{1/2}|A_l(s)|^2\frac{P_l[-1-(t/2s)]}{\sin\pi l} \qquad (2.27)$$

$$B_-(s,t)=\tfrac{1}{2}i\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dl \times(2l+1)C_l(s)P_l[-1-(t/2s)]/\sin\pi l. \qquad (2.28)$$

Convergence of (2.27) and (2.28) is again established as before, through the use of (2.16) and the techniques of Appendix A. Using now the representation (2.17) for $P_l(-z)$, the last two expressions for $B_\pm(s,t)$ are seen to be quite consistent with the representation (2.18) and (2.19). This completes the discussion of simultaneous analyticity of $f(s,t)$ in the two variables s and t, viz., $f(s,t)$ is an analytic function except for the cuts:

$$(1)\ t>\beta^2,\ s>0; \quad (2)\ t>\beta^2,\ s<-\tfrac{1}{4}\beta^2. \qquad (2.29)$$

It may be mentioned that (2.27) tacitly assumes the validity of the unitarity condition for complex l. This condition is, in the language of Fivel,[20]

$$S^*(l^*,s^{1/2})S(l,s^{1/2})=1, \qquad (2.30)$$

where

$$S(l,s^{1/2})=1+2is^{1/2}A_l(s), \quad (\mathrm{Re}s>0). \qquad (3.21)$$

Now using the definition of $D_l(s)$ given by (2.7) which is valid for $\mathrm{Re}l>-3/2$, it is seen that

$$D_l(s+i\epsilon)-D_l(s-i\epsilon)=-2is^{1/2}N_l(s), \qquad (2.32)$$

so that (2.31) is re-expressible as

$$S(l,s^{1/2})=D_l(s-i\epsilon)/D_l(s+i\epsilon), \qquad (2.33)$$

thus explicitly verifying (2.30).

[18] It has been shown by Cheng (to be published) that $A(s,t)$ is not analytic in t. This however, does not affect the analyticity of $f(s,t)$, for which it is enough that $A(s,t)$ be defined along the line $0<t<\infty$, according to (2.18).

[19] In order to ensure that the left-hand cut indeed exists, and that there are no fortuitous cancellations in the series (2.21) so as to make $B_-(s,t)$ identically zero, it is enough to show that $B_-(s,t)$ does *not* vanish over at least a certain region of s and for a small value of λ, such that terms of order (λ^3) are negligible. To order λ^2, (2.21) and (2.22) reduce to

$$B_-(s,t)\approx\frac{8\pi^3\lambda^2}{s}\sum_l(2l+1)P_l\left(1+\frac{t}{2s}\right)P_l\left(1+\frac{\beta^2}{2s}\right) \times\int_0^\infty\frac{ds'\,Q_l(1+\beta^2/2s')}{(s')^{1/2}(s'-s)}.$$

This series is convergent for $0<t<\beta^2$, according to the results of Appendix A. If we now take a *large* and *negative* value of s ($|s|\gg\frac{1}{4}\beta^2$), it is immediately seen that each term of this series is a positive quantity [since $P_l(1)=+1$ and $Q_l(x)$ for $x>1$ and integral l, has no zeros], so that the series as a whole is a nonzero positive quantity. Therefore, $B_-(s,t)$ for $s<-\frac{1}{4}\beta^2$ is not identically zero, showing that the left-hand s cut in $f(s,t)$ represents a genuine feature of our model.

[20] D. Fivel, Phys. Rev. **125**, 1085 (1962).

The additional cut over the region $t>0$, $s<0$ for the case of our potential has a simple meaning in terms of the third variable u defined by $u=-2s(1+\cos\theta)$, i.e.,

$$u+t+4s=0. \qquad (2.34)$$

Thus, $t>0$, $s<0$ corresponds to $t>0$, $u<0$ which just stands for the cut region in the so-called "crossed channel" in the field-theoretical Mandelstam representation. Curiously, it does not represent any conventional "exchange potential" effects ($s>0$, $u>0$). In any case, the representation deduced for our potential does have a place in the field-theoretical Mandelstam representation, though it is somewhat wider than the one for a purely Yukawa amplitude.

This result may not be unexpected in view of the fact that our potential is highly nonlocal. Now it is recognized[21] that interactions of finite range in field theory produce scattering amplitudes whose properties are partly analogous to those of single partial wave amplitudes with suitable nonlocal potentials of the same range. It is, therefore, likely that the amplitudes $A_l(s)$ derived from (2.2) might equally well arise from suitable truncation schemes in a field theory, and to that extent incorporate some residual effects of the latter, of the type exhibited above.

3. REGGE POLES—THRESHOLD AND HIGH-ENERGY BEHAVIOR

In this section we shall examine the amplitude (2.5)–(2.7) for singularities in the complex l plane.

We note first that the denominator function $D_l(s)$ defined by (2.7) can be analytically continued to $\mathrm{Re}l\geqslant -3/2+\epsilon(\epsilon>0)$, since near $s=0$, $Q_l(1+\beta^2/2s)\sim s^{l+1}$. The form (2.7) is especially convenient for large s. For small s on the other hand, it is more convenient (for purposes of explicit evaluation) to express (2.7) in an alternative form based on the representation (valid for $\mathrm{Re}s>0$ and $\mathrm{Im}s=0$)

$$Q_l(1+\tfrac{1}{2}\beta^2s^{-1})=\int_{\xi_0}^{\infty}d\xi\,\xi^{-l-1}(\xi^2-2x\xi+1)^{-1/2}, \qquad (3.1)$$

$$\xi_0=x+(x^2-1)^{1/2},\quad x=1+\tfrac{1}{2}\beta^2s^{-1}. \qquad (3.2)$$

Substituting (3.1) in (2.7) and interchanging the order of ξ and s' integrations, it is easy to deduce

$$D_l(s)=1-4\pi^2\lambda\int_1^{\infty}d\xi\,\xi^{-l-1}[\beta^2\xi-s(\xi-1)^2]^{-1/2}. \qquad (3.3)$$

Equation (3.3) shows, like Eq. (2.7), that $D_l(s)$ is an analytic function of s, with a cut only for real and positive s, and the discontinuity in (3.3) works out as

$$\begin{aligned}D_l(s+i\epsilon)-D_l(s-i\epsilon)&\\&=-8\pi^2\lambda i\int_{\xi_0}^{\infty}d\xi\,\xi^{-l-1}[\beta^2\xi-s(\xi-1)^2]^{1/2}\\&=-2is^{1/2}N_l(s),\end{aligned} \qquad (3.4)$$

according to (2.6) and (3.1). Thus, (2.7) and (3.3) are equivalent representations of $D_l(s)$ within their common domain of analyticity in the l plane, viz., $\mathrm{Re}l>-1$. Now the position of the Regge pole for a given s is the solution of the equation

$$D_l(s)=0,$$

so that (2.7) or (3.3) shows that $l=\alpha(s)$ is an analytic function of s for $\mathrm{Re}s<0$, and that it has a right-hand cut along the line $\mathrm{Re}s\geqslant 0$, $\mathrm{Im}s=0$.

To find the position of the Regge pole near threshold, the integration in (3.3) when $s\to 0$ is immediately performed to give

$$D_l(0)=1-4\pi^2\lambda\beta^{-1}(l+\tfrac{1}{2})^{-1}. \qquad (3.4\text{a})$$

Thus, the Regge pole at threshold is given by

$$l=\alpha(0)=-\tfrac{1}{2}+\sigma,\quad \sigma=4\pi^2\lambda/\beta. \qquad (3.5)$$

Further, it is shown in Appendix B that the D_l function near threshold has the form

$$\begin{aligned}D_l(s)=1-\sigma[1-(-s/\beta^2)^{l+\frac{1}{2}}\pi^{-1/2}&\\ \times\Gamma(l+1)\Gamma(\tfrac{1}{2}-l)]/(l+\tfrac{1}{2}).&\end{aligned} \qquad (3.6)$$

The zeros of (3.6) would, therefore, give the positions of the Regge poles near threshold. For this purpose one sets in (3.6)

$$l\equiv\alpha(s)=-\tfrac{1}{2}+\sigma+\eta(s) \qquad (3.7)$$

where, according to (3.5), $\eta(s)$ is expected to be small. Thus, $\eta(s)$ is a solution of the equation

$$\eta+(-s/\beta^2)^{\sigma+\eta}\sigma\pi^{-1/2}\Gamma(1-\sigma-\eta)\Gamma(\tfrac{1}{2}+\eta+\sigma)=0. \qquad (3.7\text{a})$$

For $\sigma>0$, this gives a solution consistent with the smallness of η[22]:

$$\eta(s)=-g(\sigma)(-s/\beta^2)^{\sigma}, \qquad (3.8)$$

$$g(\sigma)=-\pi^{-1/2}\sigma\Gamma(1-\sigma)\Gamma(\tfrac{1}{2}+\sigma). \qquad (3.8\text{a})$$

This formula, however, does not hold if σ is a positive integer, and one has to turn to Eq. (3.7a) for a solution in such a case. That is, if $\sigma=1$, Eq. (3.7a) is expressible as

$$\eta^2=\pi^{-1/2}(-s/\beta^2)^{1+\eta}\Gamma(1-\eta)\Gamma(\tfrac{3}{2}+\eta), \qquad (3.9)$$

whence one has the solution (for $\sigma=1$)

$$\begin{aligned}\eta(s)\approx-(-s/2\beta^2)^{1/2}+\tfrac{1}{4}(-s/\beta^2)&\\ \times[\ln(-s/\beta^2)+2+\psi(\tfrac{1}{2})-\psi(1)].&\end{aligned} \qquad (3.10)$$

where

$$\psi(x)=\frac{d}{dx}\ln\Gamma(x). \qquad (3.10\text{a})$$

As would be expected, Eqs. (3.8) and (3.10) show $l=\alpha(s)$ as an analytic function of s with a right-hand cut starting at $s=0$. In addition, as the origin is approached from the negative real axis of s, $\eta(s)$, while

[21] M. Ruderman and S. Gasiorowicz, Nuovo Cimento **8**, 861 (1958).

[22] It may be noted that the solution (3.8) is *not* consistent for $\sigma<0$, since η then becomes infinite for small s.

remaining real, *increases* towards zero. Finally, for $\text{Re}s>0$ and $0<\sigma\leqslant 1$, $\eta(s)$ acquires a positive imaginary part according to the replacement $s=k^2+i\epsilon$, in (3.8) or (3.10).[23] For the case $0<\sigma<1$, the real and imaginary parts of $\eta(s)$ above threshold are given, according to (3.8), by

$$\begin{aligned}\text{Re}\eta(k)&=-g(\sigma)\cos\pi\sigma\,(k/\beta)^{2\sigma},\\ \text{Im}\eta(k)&=g(\sigma)\sin\pi\sigma\,(k/\beta)^{2\sigma}.\end{aligned}\tag{3.11}$$

Both these functions have cusp behavior at $k=0$ for $\sigma<\frac{1}{2}$:

$$\lim_{k\to 0}|d\eta/dk|\to\infty,\quad(\sigma<\tfrac{1}{2}).\tag{3.12}$$

The physical meaning of the case $\sigma=\frac{1}{2}$ [which incidentally makes $\alpha(0)=0$ in (3.5)], is linked directly with the condition of bound state formation for s waves. This fact may also be checked through the old-fashioned effective-range formulas which can be deduced exactly from (2.2) for various partial waves. In particular, for s and p waves the relevant formulas are[11]

$$\begin{aligned}k\cot\delta_0&=\text{Re}D_0(k^2)/N_0(k^2)\\ &=(k^2/\beta\sigma)[1-\beta\sigma k^{-1}\tan^{-1}(2k/\beta)]/Q_0(1+\tfrac{1}{2}\beta^2k^{-2})\end{aligned}\tag{3.13}$$

$$\begin{aligned}k^3\cot\delta_1&=k^2\,\text{Re}D_1(k^2)/N_1(k^2)\\ &=\frac{k^4}{\beta\sigma}\left[1-\frac{\beta\sigma}{k}\left(1+\frac{\beta^2}{2k^2}\right)\tan^{-1}\left(\frac{2k}{\beta}\right)+\frac{\beta^2\sigma}{k^2}\right]\Big/Q_1(1+\tfrac{1}{2}\beta^2k^{-2}),\end{aligned}\tag{3.14}$$

from which the conditions of s- and p-wave bound state formation are, respectively, deducible as

$$\sigma>1/2,\quad\sigma>3/2.\tag{3.15}$$

It can be verified from (3.13) that if $\sigma<\frac{1}{2}$, i.e., an s-wave bound state does not occur, then the numerator of (3.13) does *not* vanish for $k>0$ (no s-wave resonance). On the other hand, if $3/2>\sigma>1/2$, then an examination of (3.14) shows that a p-wave resonance is possible. Precisely the same conclusion is indicated in terms of the Regge trajectories: The curve for $\text{Re}\eta$ versus k moves up (no cusp) or bends down (cusp) at threshold according as $\sigma>$ or $<\frac{1}{2}$, respectively. Thus, the trajectory (3.11) follows a pattern expected of any reasonable kind of potential.

Recently, Desai and Newton[24] have shown that the threshold behavior of a partial wave amplitude is connected with the existence of an infinite number of Regge trajectories that arrive at $l=-\frac{1}{2}$ as $s\to 0$. We would like to point out that the amplitude considered here admits of these Desai-Newton poles. For this it is necessary to examine the zeros of (3.6) near $l=-\frac{1}{2}$, so that setting $l+\frac{1}{2}=l'$, the poles near $l'=0$ are given by

$$1-g(l')(-s/\beta^2)l'=0,\tag{3.16}$$

where the function $g(x)$ is defined as in (3.8a), and $g(0)=1$.[25] From (3.16) one shows by proceeding exactly as in reference 24 that the Desai-Newton poles are given, for $s=0+$, by

$$l'\approx n\pi(\pi a^{-1}+i)/a,\tag{3.17}$$

and for $s=0-$, by

$$l'=\pi n(i-\tfrac{1}{2}A^2n\pi a^{-2})/a,\tag{3.18}$$

where

$$a=\ln|\beta^2/s|\gg 1,\tag{3.19}$$

and n is a positive integer such that $n\ll a$. The quantity A in (3.18) is defined, as in reference 24, by $g^{-1}(l')\approx 1+Al'$, so that

$$A=-g'(0)=\psi(1)-\psi(\tfrac{1}{2})-\sigma^{-1}=\ln 2+1-\sigma^{-1}.\tag{3.20}$$

The significance of n as a *positive* integer is that for the poles (3.17) and (3.18) to exist, one should have $\text{Re}l'\geqslant 0$, to allow a zero to develop in the left-hand side of (3.16) as $s\to 0$. Thus, these poles exist only on the right half of the l' plane. On the other hand, if $\text{Re}l'<0$, Eq. (3.16) cannot be satisfied unless $g(l')=0$, and this does *not* occur in the present case, as may be seen from (3.8a). Thus, the "threshold poles" of Desai and Newton are not present on the left-hand half of the l' plane in our model.

Finally, for the high-energy behavior of the principal Regge trajectory according to our potential, we turn once again to the definitions (2.7) and (3.3) of $D_l(s)$. Equation (3.3) shows that for large $|s|$ the integral term is of order $s^{-1/2}$, except for $l=-1$ when the integral diverges at the upper limit. [The same conclusion also follows, less directly, from (2.7).] At high energies one, therefore, expects the pole to shift near $l=-1$. For a more precise location one writes $l=-1+\nu$, where $|\nu|\ll 1$, and makes use of the following approximation deducible from Erdélyi[26]:

$$Q_{-1+\nu}(x)\approx\nu^{-1}-\ln\tfrac{1}{2}(x-1)-Q_0(x)+O(\nu).\tag{3.21}$$

Using (3.21) in (2.7), elementary integrations are encountered to deduce (for $\text{Re}s<0$)

$$\begin{aligned}D_l(s)\approx{}&1-\beta\sigma\nu^{-1}(-s)^{-1/2}\\ &+\beta\sigma(-s)^{-1/2}\ln\{\beta[\beta+2(-s)^{1/2}]/(-4s)\}.\end{aligned}\tag{3.22}$$

The motion of the principal Regge pole for large $|s|$ is then given by

$$\begin{aligned}l\equiv\alpha(s)=-1+\nu\approx{}&-1+\beta\sigma(-s)^{-1/2}\{1-\beta\sigma(-s)^{-1/2}\\ &\times\ln[-4s/\beta(\beta+2(-s)^{1/2})]\}^{-1},\end{aligned}\tag{3.23}$$

[23] This is, of course, the expected behavior for a Regge pole, and the sign ambiguity in η in going from (3.9) to (3.10) was resolved by this criterion.

[24] B. Desai and R. G. Newton (to be published).

[25] Our notation differs from that of reference 24 in the replacements $\lambda\to l'$ and $C(\lambda)\to g(l')$.

[26] *Bateman Manuscript Project; Higher Transcendental Functions*, edited by A. Erdélyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I.

whose analytic continuation for $s=k^2+i\epsilon$ is[27]

$$\alpha(k^2+i\epsilon)\approx -1+i\beta\sigma k^{-1}\{1-\pi\beta\sigma k^{-1} +\sigma\beta k^{-1}\tan^{-1}(2k/\beta)+i\beta\sigma k^{-1} \times\ln[\beta(\beta^2+4k^2)^{1/2}/4k^2]\}^{-1}. \quad (3.24)$$

Equations (3.23) and (3.24) show indeed that $\alpha(s)$ tends to -1 as $|s|\to\infty$.

It has by now become intuitively obvious that for the potential (2.2) we are considering, there is only *one* Regge pole to the *right* of the line $\mathrm{Re}l=-3/2$, for a given value of the coupling constant. In particular, it is the principal pole whose high-energy end is described by Eqs. (3.23)–(3.24), and whose threshold behavior is governed by Eqs. (3.8)–(3.11). This pole is, of course, distinct from the Desai-Newton poles given by (3.17) and (3.18). As for the other Regge poles, lying to the left of the line $\mathrm{Re}l=-3/2$, we are not entitled to discuss them on the basis of the representations (2.5)–(2.7) or (3.3).

For the sake of completeness we may record the high-energy limit of the amplitude defined by (2.14). For this purpose the line integral is shifted to the left up to the line $\mathrm{Re}l=-3/2+\epsilon$, according to Mandelstam's[28] prescription (cf., reference 6), viz.,

$$P_l(-z)\csc\pi l=\pi^{-1}\csc\pi l \times Q_l(-z)-\pi^{-1}\sec\pi l Q_{-l-1}(-z), \quad (3.25)$$

and the full amplitude (without now separating the Born term) is deduced as

$$f(s,t)=-\tfrac{1}{2}i\int_{-\frac{3}{2}-i\infty}^{-\frac{3}{2}+i\infty} dl\,(2l+1)A_l(s)/\pi\cos\pi l -\pi^{-1}\sum_{n=1}^{\infty}(-1)^{n-1}2nA(s,n-\tfrac{1}{2})Q_{n-\frac{1}{2}}(z) +(2\alpha+1)\sec\pi\alpha Q_{-\alpha-1}(-z)f_\alpha(s), \quad (3.26)$$

where

$$f_\alpha(s)=\left[N_l(s)\Big/\frac{d}{dl}D_l(s)\right]_{[l=\alpha(s)]}. \quad (3.27)$$

In the limit of high-momentum transfer ($|z|\gg 1$), the amplitude (3.26) reduces in the standard way to its last term. For algebraic simplicity we consider only the case

$$s=k^2+i\epsilon,\quad |t/s|\gg 1,\quad\text{and}\quad |s/\beta^2|\gg 1. \quad (3.28)$$

In this limit it is easily verified that the following results hold:

$$\alpha(s)=-1+\nu\approx -1+i\beta\sigma s^{-1/2}; \quad (3.29)$$

$$Q_\alpha(1+\tfrac{1}{2}\beta^2 s^{-1})\approx\left[\frac{d}{dl}D_l(s)\right]_{l=\alpha(s)}\approx\nu^{-1}\approx -is^{1/2}/\beta\sigma; \quad (3.30)$$

$$Q_{-\eta}(-1-\tfrac{1}{2}ts^{-1})\approx 2(-t/s)^{-1+\nu}. \quad (3.31)$$

Substitution of these results in (3.26) gives in this limit

$$f(s,t)\approx -2\beta\sigma t^{-1}(-t/s)^{i\beta\sigma s^{-1/2}}. \quad (3.32)$$

Changing s to t and vice versa in (3.32) we obtain for the high-energy amplitude $\tilde{f}(s,t)$ in the crossed channel

$$\tilde{f}(s,t)\approx\gamma(t)s^{\alpha(t)}, \quad (3.33)$$

where, consistently with (3.29),

$$\alpha(t)=-1+i\beta\sigma t^{-1/2} \quad (3.34)$$

and

$$\gamma(t)=-2\beta\sigma(-t)^{-i\beta\sigma t^{-1/2}} \quad (3.35)$$

$$\approx -2\beta\sigma[1-i\beta\sigma t^{-1/2}\ln(-t)+\cdots] \quad (3.36)$$

for small σ.

These results are completely in accord with the expected behavior of the high-energy amplitudes from Yukawa-type potentials [see reference 6].

4. DISCUSSION

We are now in a position to give a detailed assessment of our results in relation to Yukawa potential scattering. It is clear that our potential has many properties analogous to a Yukawa potential of range β^{-1}. The Regge trajectories as well as the high-energy behavior of the scattering amplitude follow closely the Yukawa pattern. The complete amplitude $f(s,t)$ is analytic in both t and s, having a bounded behavior as $t\to\infty$, and exhibiting cuts for (1) $t>\beta^2$, $s>0$ and (2) $t>\beta^2$, $s<-\frac{1}{4}\beta^2$. This second cut, however, has no analog in pure Yukawa potential scattering, and corresponds to the so-called "crossed channel" (t, u variables). The amplitude thus seems to exhibit some features characteristic of "field-theoretical" amplitudes. It has come to the authors' notice that a similar conclusion about the existence of a cut $t>t_0$, $s<-s_0$ using a different type of separable potential was reached by Cushing.[29]

A closer comparison of our results with the "Mandelstam representation" for Yukawa potential scattering, however, reveals the following points of dissimilarity with the Yukawa case. The Yukawa amplitude has additional branch cuts starting successively at the points $t=n^2\beta^2$ ($n=2, 3, 4, \cdots$), and the discontinuities across these successive cuts are given by the standard prescription of analyticity and unitarity.[5,20] This important manifestation of the Mandelstam representation is not present in our model given by Eqs. (2.2) and (2.3). Thus our amplitude does *not* show the points $t=n^2\beta^2$ as thresholds of any fresh discontinuities, the discontinuity functions at these points being merely the continuation of the one starting at $t=\beta^2$. We would like to point out however, that the absence of the above feature is just a consequence of the very special kind of potential chosen, viz., one which gives a left-hand cut in $A_l(s)$ corresponding to the *first* Born approximation for

[27] Note again that $\alpha(k^2+i\epsilon)$ has a positive imaginary part, as required by theory.

[28] S. Mandelstam, Ann. Phys. (N. Y.) **19**, 254 (1962).

[29] J. T. Cushing, Ph.D. thesis, Iowa State University, 1963 (unpublished).

N_l in the N/D solution. On the other hand, the successively higher cuts at $t=n^2\beta^2 (n=2,3,4,\cdots)$ in a Yukawa amplitude can be most easily understood in terms of the higher Born approximations to the amplitude, the nth Born approximation showing for the first time the branch point at $t=n^2\beta^2$ (see reference 5). Since, however, the effects of these successive Born approximations are absent in our model, the t-discontinuity function of our model falls short of the corresponding function in a Yukawa amplitude by roughly the contributions from (1) the second Born approximation in the interval $4\beta^2\leqslant t\leqslant 9\beta^2$, (2) the second and third Born approximations in $9\beta^2\leqslant t\leqslant 16\beta^2$, and so on.

From these considerations, a possible way of extending our potential so as to incorporate the additional branch points at $t=n^2\beta^2$ is suggested on the following lines. In Eq. (2.5) of reference 1 a prescription was given for writing down an "equivalent separable potential" which would give the same first Born amplitude as the actual potential in question. In a similar way, a separable potential could be written down so as to include the contribution of the second Born Yukawa amplitude in its definition, viz.,

$$v_l^2(p)=(8\pi^3\lambda)^{-1}\int d\Omega\, P_l(\cos\theta)f^{(2)}(k^2,\cos\theta), \quad (4.1)$$

where $f^{(2)}$ is the Yukawa amplitude *up to* the second Born approximation and (4.1) replaces (2.3). This separable potential is not quite "equivalent" to the Yukawa potential in the above sense of reproducing the second Born amplitude up to $O(\lambda^2)$, since a λ^2 contribution from the first-order N/D solution would also be included in the result. However, (4.1), from its very definition, would now include the effect of the additional t cut for $t\geqslant 4\beta^2$. The additional discontinuity across this cut would be automatically incorporated in the formalism described in Sec. 2, since unitarity for each partial wave is explicitly built into this formalism. However, the mathematical structure of the amplitude would not be amenable to a "prescription" of the type discussed by Blankenbecler *et al.*,[5] wherein the discontinuity function is built up in successively larger regions $t>n^2\beta^2$ through a knowledge of this function for $t<n^2\beta^2$. A similar extension to include the t cuts for $9\beta^2$, $16\beta^2$, etc., could be made on the lines of Eq. (4.1) though the procedure would fast get extremely cumbrous. It does, however, give some indication of the type of contributions that are missing from the potential (2.3) compared with the Yukawa potential, and the type of terms needed to compensate for them. Later in this section we shall give a more exact (but formal) definition of the "equivalent separable potential" than is provided by Eq. (2.5) of reference 11, or Eq. (4.1) of this section.

A somewhat different insight into the meaning of our separable potential (2.2) and (2.3) comes from a comparison of the amplitude $A_l(s)$ with the so-called Fredholm solutions to the Yukawa amplitudes for each partial wave which have been given recently by Lee and Sawyer.[30] It is immediately seen that $A_l(s)$ is just the amplitude corresponding to a truncation to first order in λ in *both* the numerator and denominator of the Fredholm solution. This feature is not unexpected since a separable potential automatically leads to solutions in which the Fredholm denominator terminates at a finite integral power λ^n (in the present case, $n=1$). However, this correspondence of our result with the first-order truncated Fredholm amplitude gives a clue to the appearance of the left-hand s cut in our $f(s,t)$. The qualitative reasoning is roughly as follows. Since we know from earlier work[5] that the exact Yukawa amplitude $f_Y(s,t)$ does not have a left-hand s cut, it is clear that the same must be true when the *complete* Fredholm amplitude is considered. Our result of Sec. 2 indicates, on the other hand, that a truncated Fredholm amplitude can show a residual left-hand cut. These two statements can be quite consistent with each other when we recognize the possibility of "cancellations" within a Fredholm amplitude of a certain order of truncation. Thus, for small values of λ, the first-order Fredholm truncation shows a left-hand cut of order (λ^2), as well as a right-hand cut of order λ^2. However, in a higher order of truncation say the nth, it is more likely that the left-hand cut is (for small λ) of order λ^{n+1}, while the right-hand cut still remains of order λ^2. It is just fortuitous that for our case of $n=1$, the left- and right-hand cuts appear to be of the same order of magnitude, but the difference between them must show up in successively higher orders of truncation in order that in the limit of $n\to\infty$, the left-hand cut may vanish altogether.

Finally it may be of some interest to compare our amplitude (2.5) with one obtained by taking only the Born approximation to the left-hand cut in the various partial wave amplitudes.[5] These amplitudes $f_l(s)$ are defined for the present case by the equation

$$f_l(s)=N_l(s)+\int_0^\infty ds'\, s'^{1/2}|f_l(s')|^2/(s'-s), \quad (4.2)$$

where $N_l(s)$ is given by (2.6). The amplitude $f_l(s)$, by its very definition, leads only to the right-hand cut in s in the full amplitude $f_1(s,t)$. This is seen by summing over all the partial amplitudes according to (2.3) and noting that the first term in (4.1) gives just the term (2.12). The integral term in (4.2), on the other hand, has only a right-hand cut in the s variable. In the notation of Eq. (2.26), the amplitude $f_1(s,t)$ has the representation

$$f_1(s,t)=8\pi^2\lambda(\beta^2-t)^{-1}+\int_0^\infty ds'\, B_+'(s',t)/(s'-s), \quad (4.3)$$

where $B_+'(s,t)$ is given by (2.27) with $A_l(s)$ replaced by $f_l(s)$, and the corresponding $B_-'(s,t)$ is zero in (4.2).

[30] B. W. Lee and R. F. Sawyer, Ann. Phys. (to be published).

In a formal way, it is possible to modify the potential (2.2) so as to lead only to a representation like (4.3). One could always write the solution of (4.2) in the form

$$f_l(s)=N_l'(s)/D_l'(s), \quad (4.4)$$

with $N_l'(s)$ and $D_l'(s)$ having only left-hand and right-hand cuts, respectively,[31] and then define a new separable potential $v_l'(p)$ through an equation of the form [cf., (2.2)]

$$v_l'^2(p)=(2\pi^2\lambda)^{-1}N_l'(p^2). \quad (4.5)$$

This definition is of course a highly implicit one insofar as $N_l'(s)$ and $D_l'(s)$ satisfy two coupled integral equations. However, a formal definition like (4.5) of the "equivalent separable potential" is quite unambiguous and works equally well for any given amplitude [not necessarily (4.2)], expressed in an N/D form. For a two-body system, a separable potential defined in the above manner—starting from a given amplitude $A(s,t)$ and projecting out its various partial waves $f_l(s)$—is in some sense equivalent to the generalized two-body potential of Chew and Frautschi[32] for the *same* amplitude $A(s,t)$ [the connecting link being provided by the common amplitude $A(s,t)$]. For a three-particle system, on the other hand, the predictions may be different for the two cases, and as has been pointed out already in Sec. 1, separable potentials have at least a computational advantage for a three-particle system.

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APPENDIX A

For all z outside $-\infty<z<1$ and Re$l-1/2$, the asymptotic form of $Q_l(z)$ for large $|l|$ is

$$Q_l(z)\approx(\pi/2)^{-1/2}(z^2-1)^{-1/4}[z-(z^2-1)^{1/2}]^{l+1} \times\Gamma(l+1)/\Gamma(l+\tfrac{3}{2}). \quad (A1)$$

In particular, this formula holds for positive integral l. Substituting (A1) in (2.7) and making the transformation

$$1+(\beta^2/2s')=\cosh[2y/(2l+1)],$$

one finds for $l\gg1$ (positive integer)

$$1-D_l\approx4\lambda\left(\frac{\pi}{2l+1}\right)^{3/2}\int_0^\infty dy\,\beta e^{-y}\left(2\coth\frac{y}{2l+1}\right)^{1/2}\Bigg/\left[\beta^2-4s\sinh^2\left(\frac{y}{2l+1}\right)\right]. \quad (A2)$$

Since most of the contribution to the integral comes from $y\lesssim1$ [due to the presence of $\exp(-y)$], it is possible to neglect $y/(2l+1)$ compared with unity, so that for s not too large, (A2) is evaluated asymptotically to yield

$$1-D_l\approx8\pi^2\lambda\beta^{-1}(2l+1)^{-1} \times\{1+O[(2l+1)^{-1},\,s\beta^{-2}(2l+1)^{-1}]\}. \quad (A3)$$

Thus, for l sufficiently large, one has $D_l\approx1$, so that $A_l\approx N_l$. Further, by setting for $s=k^2+i\epsilon$ ($\epsilon>0$),

$$1+(\beta^2/2s)=\cosh(\alpha-i\epsilon'),\quad(\alpha>0), \quad (A4)$$

(A1) can be expressed as

$$Q_l(1+\beta^2/2s)\approx[\pi/(2l+1)\sinh\alpha]^{1/2} \times\exp[-(\alpha-i\epsilon')(l+\tfrac{1}{2})], \quad (A5)$$

where $\epsilon'>0$, by virtue of (A4). On the other hand, the asymptotic behavior, for large $|l|$ ($l=l_R+il_I$), of $P_l(1+t/2s)$ occurring in (2.3) is given by[10]

$$P_l(1+t/2s)\approx[\pi(2l+1)\sinh\zeta]^{-1/2} \times[e^{(l+\frac{1}{2})\zeta}\pm ie^{-(l+\frac{1}{2})\zeta}], \quad (A6)$$

where

$$z=1+(t/2s)=\cosh(\zeta)=\cosh(\xi+i\eta),\quad \xi\geqslant0, \quad (A7)$$

and the $\pm$ signs in (A6) correspond, respectively, to Im$z>$ or <0. Thus

$$P_l(1+t/2s)\lesssim|\pi(2l+1)\sinh\zeta|^{-1/2} \times\exp\{\pm[\xi(l_R+\tfrac{1}{2})-\eta l_I]\}, \quad (A8)$$

the upper or lower signs being taken according to the criterion of a positive exponent. For positive integral l ($l_I=0$), substitution of (A5) and (A8) in (2.3) leads, therefore, to the condition of convergence as

$$\xi<\alpha, \quad (A9)$$

which is just the Lehmann ellipse and corresponds, for $\epsilon'=\eta=0$, to

$$0<t<\beta^2. \quad (A10)$$

It may be that the series (2.3) is also convergent for s on the real axis (negative side) of the s plane. Indeed, In Eq. (A4), if $s=-k_1^2$ (real), where $k_1^2<(1/4)\beta^2$, one has

$$-x=1-(\beta^2/2k_1^2)=-\cosh\alpha_1,\quad \alpha_1>0. \quad (A11)$$

Further, since for integral l,

$$Q_l(-x)=(-1)^{l-1}Q_l(x), \quad (A12)$$

Eq. (A1) can now be used to evaluate $Q_l(-x)$, via (A12). For s lying on the real axis between 0 and $-(1/4)\beta^2$, the Q_l function in (2.3), therefore, yields a convergent exponential factor as in (A5). Thus, following the steps from (A5) to (A10), one again arrives (for

[31] H. P. Noyes and D. Y. Wong, Phys. Rev. Letters **3**, 191 (1959).

[32] G. F. Chew and S. C. Frautschi, Phys. Rev. **124**, 264 (1961).

t real and positive) at precisely the condition (A10), as long as $k_1^2 < t/4$, i.e., $s > -t/4$. On the other hand, for $s < -t/4$, (A7) gives $\xi = 0$, implying from (A6) that $P_l(1+t/2s)$ has only an oscillatory behavior in l (exponent purely imaginary). Thus the convergence of the series (2.3) is assured for negative real s up to $-\beta^2/4$. For values of s with $\mathrm{Re}s < -\beta^2/4$, even the Q_l function in (2.3) has an oscillatory exponential factor, since now $1+(\beta^2/2s)$ is numerically less than unity (for $\mathrm{Im}s = 0$). The convergence of the series in this case is facilitated by separating out from $A_l(s)$ its Born approximation part $N_l(s)$, according to Eqs. (2.9)–(2.11) of the text, so that from (A3) the quantity $B_l(s)$ of (2.9) picks an additional factor $(2l+1)^{-1}$ compared with $A_l(s)$. The convergence of the series

$$\sum_l (2l+1) B_l(s) P_l(z), \tag{A13}$$

now follows for $s = -k_2^2$ (real) and $0 < t < \beta^2$, where $k_2^2 > \beta^2/4$, in the same way as before. It may be noted that as a function of s, $P_l(1+t/2s)$ has no cuts for $s < -\beta^2/4 < -t/4$. However, since the Q_l function that appears in $B_l(s)$ has cuts on the negative real axis for $s < -\beta^2/4$, one must use $s = -k_2^2 \pm i\epsilon$ in its argument, to ensure proper analytic behavior in s. By this procedure one has

$$1+\tfrac{1}{2}\beta^2(-k_2^2 \pm i\epsilon) = \cosh(\epsilon_2 \mp i\eta_2), \tag{A14}$$

where $\epsilon_2 > 0$ and $0 < \eta_2 < \pi$. Eq. (A5) should now be replaced by

$$Q_l[1+(\beta^2/2s)] \approx [\pi/(2l+1)\sinh(\epsilon_2 \mp i\eta_2)]^{1/2} \times \exp[-(l+\tfrac{1}{2})(\epsilon_2 \mp i\eta_2)]. \tag{A15}$$

Explicitly,

$$\epsilon_2 \approx (\epsilon\beta^2/2k_2^4)\csc\eta_2, \quad \text{and} \quad \csc\eta_2 = 1-\beta^2/2k_2^2. \tag{A16}$$

APPENDIX B

Consider the integral

$$I_l(s) = 1 - D_l(s) = \sigma\beta \int_0^1 d\xi\, \xi^l [\beta^2\xi - s(\xi-1)^2]^{-1/2} \tag{B1}$$

in the region of real $s<0$, s being small. Setting $s = -y\beta^2$, where $0 < y \ll 1$, (B1) can be divided into the regions

$$(1)\ 0 \leqslant \xi \leqslant y; \quad (2)\ y \leqslant \xi \leqslant 1. \tag{B2}$$

In region (1), $y(1-\xi)^2 \approx y$, so that

$$I_l^{(1)} \approx \sigma \int_0^y d\xi\, \xi^l (\xi+y)^{-1/2}$$

$$= \sigma y^{l+\frac{1}{2}} \int_0^1 d\xi\, \xi^l (\xi+1)^{-1/2} \quad (\text{changing } \xi \text{ to } \xi y). \tag{B3}$$

This integral is convergent as long as $\mathrm{Re}l > -1$. In region (2),

$$I_l^{(2)} = \sigma \int_y^1 d\xi\, \xi^l [\xi + y(1-\xi)^2]^{1/2}$$

$$= \sigma \int_y^1 d\xi\, \xi^{l-\frac{1}{2}} \left[1 + \sum_1^\infty \binom{-\frac{1}{2}}{n} y^n \xi^{-n} (1-\xi)^{2n}\right].$$

The integration can now be performed by observing that the contribution of the upper limit in *all* the terms of the n summation vanishes, and that in the lower limit (y), $(1-\xi)^2$ is well approximated by unity (since $y \ll 1$). This gives, after a slight rearrangement,

$$I_l^{(2)} \approx \sigma \left[\frac{1}{l+\frac{1}{2}} + \sum_0^\infty \binom{-\frac{1}{2}}{n} \frac{y^{l+\frac{1}{2}}}{n-l-\frac{1}{2}}\right]. \tag{B4}$$

Using the identity

$$\sum_0^\infty \binom{-\frac{1}{2}}{n} \frac{1}{n-l-\frac{1}{2}} = \int_0^1 d\xi\, \xi^{-l-\frac{3}{2}} (1+\xi)^{-1/2} = \int_1^\infty d\xi\, \xi^l (\xi+1)^{-1/2}, \tag{B5}$$

valid for $\mathrm{Re}l < -1/2$, one finds from (B3)–(B5)

$$I_l(s) = I_l^{(1)} + I_l^{(2)}$$

$$\approx \sigma \left[(l+\tfrac{1}{2})^{-1} + y^{l+\frac{1}{2}} \int_0^\infty d\xi\, \xi^l (\xi+1)^{-1/2}\right] \tag{B6}$$

$$= \sigma (l+\tfrac{1}{2})^{-1} [1 - \pi^{-1/2} y^{l+\frac{1}{2}} \Gamma(l+1) \Gamma(\tfrac{1}{2}-l)], \tag{B7}$$

noting the relation

$$\int_0^\infty d\xi\, \xi^l (\xi+1)^{-1/2} = \Gamma(l+1)\Gamma(-l-\tfrac{1}{2})/\Gamma(\tfrac{1}{2}). \tag{B8}$$